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Technical Note

An analytical solution for a Stefan problem with variable latent heat

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Abstract

Governing equations for a one-phase Stefan melting problem with variable latent heat are presented. It is shown that these equations model the movement of the shoreline in a sedimentary basin. An analytical solution for the sedimentation rate and shoreline movement—based on a similarity variable—shows a square root dependence of shoreline position with time.

1. Introduction

The objective of this paper is to present an analytical solution of a one-phase Stefan melting problem which involves a latent heat that is a linear function of space. The governing equation is

$$\frac{\partial h}{\partial t} = v\frac{\partial^2 h}{\partial x^2} \quad 0 \leqslant x \leqslant s(t) \tag{1}$$

with boundary conditions

$$v\frac{\partial h}{\partial x}\bigg|_{x=0} = -\bar{q}, \quad \text{and} \quad h|_{x=s(t)} = 0 \tag{2}$$

where v is a diffusion coefficient, the flux $\bar{q}$ is prescribed and $s(t)$ is the moving melt interface. To close the problem, the additional balance

$$v\frac{\partial h}{\partial x}\bigg|_{x=s} = -\gamma s\frac{\mathrm{d}s}{\mathrm{d}t} \tag{3}$$

needs to be satisfied on this moving interface, where γ is a given constant. The problem (1)–(3) differs in two respects from the standard one-phase Stefan melting problem [1]:

1. A fixed-flux Neumann boundary condition is applied at the origin, $x = 0$, as opposed to the usual fixed-value Dirichlet condition.
2. On comparison of (3) with the one-phase Stefan condition (see Ref. [1]) it is seen that the latent heat term, γs, is not constant but, rather, a linear function of position.

In the context of conventional melting problems it is difficult to provide a physical rational for the study of problems where the latent heat is a function of space. The formulation in (1)–(3), in particular the condition in (3), is, however, a physically meaningful limiting problem in the study of shoreline movement in a sedimentary basin.

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Nomenclature

b	height of the Earth's crust above datum, m
h	sediment height above datum, m
$\bar{q}$	prescribed sediment line-flux, $\mathrm{m^3\,m^{-1}\,t^{-1}}$
$s(t)$	location of shoreline, m
t	time, s
$u(t)$	location of intersection between off shore sediment wedge and basement, m
x	space dimension, m
$z(t)$	ocean level above datum, m
Greek symbols	
α	slope of off shore sediment wedge –
β	basement slope –
γ	constant in modified Stefan condition, $\mathrm{mt^{-\frac{1}{2}}}$
η	scaled sediment height, $\mathrm{mt^{\frac{1}{2}}}$
λ	interface position parameter –
ν	diffusion coefficient, $\mathrm{m^2\,t^{-1}}$
ξ	similarity variable, $\mathrm{mt^{-\frac{1}{2}}}$

2. The shoreline problem

In recent work Swenson et al. [2] developed a model for the movement of a shoreline in a sedimentary basin in response to changes in sediment line-flux, tectonic subsidence of the Earth's crust, and sea level change. A schematic of the basin cross-section is shown in Fig. 1. At a given point in the process two domains can be identified in the system:

1. A subaerial, fluvial domain. In this domain, the sediment transport and deposition can be modeled by the diffusion Eq. [2]

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\partial b}{\partial t}, \quad 0 \leqslant x \leqslant s(t) \tag{4}$$

where h is the height of the sediment above a datum, b is the height of the Earth's crust (the tectonic plate), and the diffusivity ν depends on the characteristics of the sediment grains and the time-averaged water line-discharge over the fluvial surface. The boundary conditions on (4) are

$$\nu \frac{\partial h}{\partial x}\bigg|_{x=0} = -\bar{q}(t), \quad \text{and} \quad h|_{x=s(t)} = z(t) \tag{5}$$

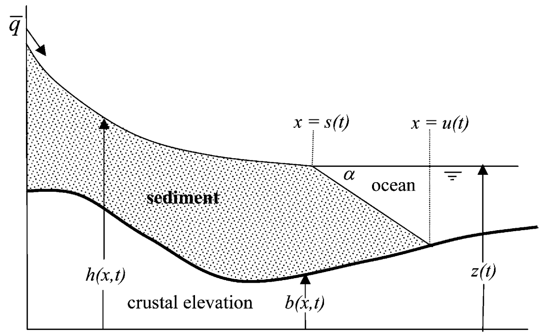

Fig. 1. A schematic cross-section of a sedimentary ocean basin.

where $\bar{q}$ is a prescribed sediment line-flux and $z(t)$ is the ocean level above the datum.

2. An offshore submarine domain. In general sediment transport in this domain is controlled by a combination of slope- and wave-current-driven processes. Swenson et al. [2] propose a simple treatment that sets the offshore sediment surface at a fixed angle of repose defined by slope α. In this treatment it is assumed that the grain movement by subaqueous avalanches is much more rapid that the movement of sediment by the fluvial processes in the subaerial domain; an assumption that has been validated by experiment [3]. In this way, the offshore can be modeled as a "sediment wedge" that is maintained by the landward supply of sediment to the shoreline. The balance of the sediment in the wedge (see Swenson et al. [2] for details) can be used to provide a condition for the advance or retreat of the shoreline, i.e.,

$$-\nu \frac{\partial h}{\partial x}\bigg|_{s(t)} = (u - s)\left[\alpha \frac{\mathrm{d}s}{\mathrm{d}t} + \frac{\mathrm{d}z}{\mathrm{d}t}\right] - \int_s^u \frac{\partial b}{\partial t}\,\mathrm{d}x \tag{6}$$

where $u(t)$ is the x position of the intersection of the sediment-wedge toe with the basement.

Eqs. (4)–(6) define the ocean-basin shoreline-tracking problem. In seeking numerical and semi-analytical solutions, Swenson et al. [2] treat the condition in (6) as a generalized Stefan condition and refer to the equation as the shoreline-Stefan condition.

3. A limit case

A limit case for the above shoreline model can be arrived at by considering a problem with a fixed sediment line-flux, a constant ocean level ($z = 0$) and no tectonic subsidence of the Earth's crust—a good approximation on many modern continental margins. If the additional assumption of a basement with constant slope $\beta < \alpha$ is made (see Fig. 2) so that by geometric construction

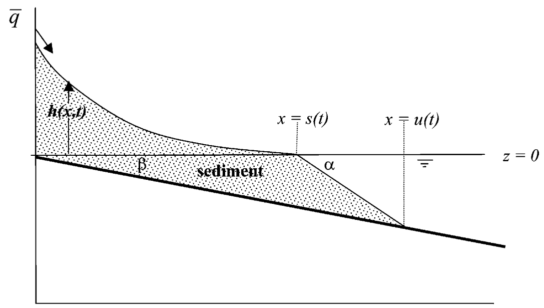

Fig. 2. Limit case with no tectonic subsidence or sea-level change.

$\alpha(u-s) = \alpha\beta s/(\alpha-\beta) = \gamma s$, the shoreline-tracking problem, defined by Eqs. (4)–(6), reduces to the variable latent heat Stefan problem given in Eqs. (1)–(3).

4. The similarity solution

On setting the movement of the shoreline to be

$$s = 2\lambda t^{\frac{1}{2}}, \tag{7}$$

introducing the similarity variable

$$\xi = \frac{x}{2t^{\frac{1}{2}}}, \tag{8}$$

and scaling the sediment height by

$$\eta = \frac{h}{2t^{\frac{1}{2}}}, \tag{9}$$

Eq. (1) and its boundary conditions (2) can be written as an ODE in $\eta(\xi)$, viz.,

$$\frac{v}{2}\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} + \xi\frac{\mathrm{d}\eta}{\mathrm{d}\xi} - \eta = 0, \quad 0 \leqslant \xi \leqslant \lambda \tag{10}$$

with

$$v\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\bigg|_{\xi=0} = -\bar{q} \quad \text{and} \quad \eta|_{\xi=\lambda} = 0 \tag{11}$$

Eqs. (10) and (11) have the solution

$$\eta(\xi) = \frac{\bar{q}}{v}\left[\lambda\left(\frac{e^{\frac{-\xi^2}{v}} + \pi^{\frac{1}{2}}v^{-\frac{1}{2}}\xi\operatorname{erf}(\xi v^{-\frac{1}{2}})}{e^{\frac{-\lambda^2}{v}} + \pi^{\frac{1}{2}}v^{-\frac{1}{2}}\lambda\operatorname{erf}(\lambda v^{-\frac{1}{2}})}\right) - \xi\right] \tag{12}$$

In the similarity variable, ξ, the shoreline Stefan condition (3) can be written as

$$v\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\bigg|_{\xi=\lambda} = -2\gamma\lambda^2 \tag{13}$$

from which, on using (12), the following non-linear equation for the constant λ is obtained

$$f(\lambda) = \frac{\pi^{\frac{1}{2}}v^{-\frac{1}{2}}\operatorname{erf}(\lambda v^{-\frac{1}{2}})}{e^{\frac{-\lambda^2}{v}} + \pi^{\frac{1}{2}}v^{-\frac{1}{2}}\lambda\operatorname{erf}(\lambda v^{-\frac{1}{2}})} - \frac{1}{\lambda} + \frac{2\gamma\lambda}{\bar{q}} = 0 \tag{14}$$

Solving (14) for λ will, on substitution into (7), provide a tracking of the shoreline $s(t)$ with time. Note that, for positive values of v, $\bar{q}$, and $\gamma(\alpha > \beta)$, $\mathrm{d}f/\mathrm{d}\lambda > 0$ for all $\lambda > 0$ and $f(\lambda) \to -\infty$ as $\lambda \to 0$ and $f(\lambda) \to \infty$ as $\lambda \to \infty$. Hence, one and only one positive value of λ will be a solution of (14).

5. Conclusions

In this short note a similarity solution for a one-phase Stefan melting problem has been presented. The novel feature in the problem is a latent heat that increases linearly with distance from the origin; a feature that can be associated with a model of shoreline movement in a sedimentary basin during a period of tectonic inactivity. The similarity solution exhibits a square root dependence of shoreline position with time.

The similarity solution in (7), (12) and (14) is worthwhile from two points of view:

1. The solution provides an explicit analytical solution that can be used to verify general computational phase change algorithms. In this respect the Neumann flux condition at $x = 0$ is outside of the norm of conditions found in existing analytical solutions of phase change problems, see [4, Chapter 10]. The only other phase change similarity solutions, known to the authors, that include Neumann conditions are due to Tarzia and co-workers [5–8]; in these solutions, however, a time dependent flux of the form $\bar{q}/\sqrt{t}$ needs to be specified to realize an explicit solution.
2. The central interest in studying shoreline motion is to understand how surface processes interact with changes in sea level to control the formation of strata in sedimentary basins. In this respect, the analytical solution of the limit shoreline model (1)–(3) provides a worthwhile benchmark for the development of numerical models that are designed to tackle more complex problems.

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